Division of fractions is a common area of difficulty for seventh graders, despite having had numerous experiences with both division and fractions in earlier grades. Teachers who anticipate limited conceptions can help students overcome the challenges they may face with this important topic. Asking students to write story problems regarding a particular mathematical concept can offer unique insights into their thinking. The ways that students decide to have characters interact with various quantities can reveal their knowledge of numbers and operations and their abilities to access each in realistic scenarios. To identify areas of concern, teachers can use problem posing to assess students’ thinking about division of fractions and plan instruction accordingly (Silver 1994). This article brings awareness to and offers instructional recommendations for addressing five areas of concern identified through an analysis of seventh graders’ student-authored division of fraction stories.

BACKGROUND

Helping students develop conceptual understanding of fraction division in a profound manner, based on abstractions of existing knowledge, is a critical, yet complicated, enterprise. Rational-number knowledge stems naturally from realistic scenarios that introduce a need for nonwhole numbers (Streefland 1991), whereas rational-number algorithms stem from generalizations of whole-number operations (Park, Güçler, and McCrory 2013). At the same time, students need to become comfortable moving back and forth between thinking about whole numbers and rational numbers (Kieren 1988) and viewing fractions in more flexible ways than afforded by the standard part-whole interpretation (Behr et al. 1983).

Unfortunately, students may feel disconnected from the quotient, operator, measurement, and ratio definitions of fractions when comparing them to their prior experiences. Similarly, K–grade 6 students tend to treat division as repeated subtraction or fair sharing. This can further complicate efforts to understand the common multiply-by-the-reciprocal algorithm.
for fraction division that is based on the inverse property of multiplication (i.e., the product of a number and its reciprocal equals one). Such issues can interfere with students being able to connect whole-number and rational-number division.

**THE PROBLEM-POSING TASK**

To open a “window into students’ mathematical understanding” (Silver 1994, p. 24), 201 seventh graders from an urban middle school in the Midwest presented evidence of their fractional thinking by writing fraction division stories. All these students had experienced operations with fractions in prior grades and had already participated in grade-level lessons on fraction addition, subtraction, and multiplication. On the third day of their fraction division sequence, the students responded to this open-ended, problem-posing task:

Write a division of fractions problem and solve it.

For their responses, students selected their own fractional values and contexts. The instructional goal was to identify conceptual limitations that students exhibited that might then give the teacher information on which to base the final fraction division lesson.

Although many students constructed well-conceived, engaging stories of division, 32 of the 201 students left the task blank (28 of whom were English language learners), and an alarming number of students wrote stories calling for fraction multiplication (23 students), subtraction (22 students), or addition (2 students). Some created sophisticated stories demonstrating precise use of fractional units, reasonable contexts, and clear wording; unfortunately, their problems did not require division. For example:

- Ashley wrote, “John had 3/4 of a cup of tea. Marcus had 2 and 1/4 cups of coffee. How much would it take John to have as much tea as Marcus had coffee?”
- Charles asked, “I had 1 2/3 quarts of milk. On Monday, I drank 1/4 of it. How much did I drink?”

By identifying underlying issues that can cause students to mistake other operations for division, teachers can generate targeted and meaningful activities designed to deepen students’ knowledge of division and the ways in which it compares with the other operations. For example, students’ familiarity with defining division as repeated subtraction might contribute to their misuse of a subtraction scenario for division. Furthermore, prior whole-number experiences with division as a “process that always made things smaller” may be confused with the fact that multiplication by a proper fraction also makes things smaller.

Whole-class discussions can thoughtfully address the relationships and differences between multiplication and division with rational numbers. Taber (2007) used this kind of approach to compare multiplication of rational numbers with both division and subtraction, highlighting the underlying structure of each operation and the way in which each asks a different question.

**IDENTIFYING LIMITED CONCEPTIONS**

All the seventh graders’ story problems that correctly used division contexts were analyzed for emergent trends of conceptual difficulties. The open-ended, problem-posing task, in which students created their own scenarios and selected their own values, revealed five areas of concern:

1. Mishandling remainders
2. Failing to label units
3. Misusing discrete fractional quantities
4. Redirecting fractional contexts to solve with whole-number amounts
5. Relying disproportionately on repeated subtraction

Examples of student work illustrating each area of conceptual difficulty are presented below, followed by instructional recommendations.

**1. Mishandling Remainders**

One of the first trends we noticed was the way in which several students carefully selected fractional values to prevent their division problems from having remainders. Although choosing fractions in this way was an exciting demonstration of number sense, it suggests that students were uncomfortable working with mixed numbers. This notion was strengthened by the number of students who demonstrated a misunderstanding of the mixed-number solutions that occurred in their problems.

For example, after successfully using the multiply-by-the-reciprocal algorithm to find a correct numerical solution, Anna was unable to appropriately interpret her answer. Instead of seeing the mixed number as one holistic value, she viewed it as a whole number with a fractional remainder (see fig. 1). Notice how she reported her answer using two different unit words: 28 days and 1/2 cup, instead of 28 1/2 days.

**Instructional recommendations:** Students like Anna may be viewing division of fractions as a completely new, unrelated concept, instead of seeing it as an extension of their experiences with whole-number division. Teachers can reinforce this connection using a series of problems that transition from (a) dividing whole numbers to produce an answer with a whole-number remainder to (b) expressing a
whole-number remainder as a fraction (with emphasis on the associated units) to (c) dividing fractions to produce interpretable answers involving a mixed number. For example, start with this problem—“McKenna has 21 friends coming to her birthday party. If 8 kids can sit together at each table, how many tables will the children fill?”—and its answer 2R5.

Labeling the answer with associated units can be beneficial for students to see that in 2R5 the 2 represents full tables and the 5 represents children. The next instructional challenge is to help students see that 5 children would fill 5/8 of another table. From here, the lesson should illustrate how mixed numbers can be used to represent the number of full tables and leftover children in a single way, using the same unit (i.e., 2 5/8 tables).

Understanding appropriate use of units is crucial to students’ interpretation of mixed-number solutions when applying algorithms to divide fractions. Anna may have interpreted her solution to the question in figure 1 more accurately if she had been asked targeted questions related to renaming her remainder. Gregg and Gregg (2007) suggest using questions such as these:

- How many full days will the food last?
- How much food will be leftover?
- Is it enough to feed Happy for another whole day?
- How about a partial day?
- If so, what portion of the next day could you feed Happy?

It is also important for students to understand that in some contexts, a mixed-number answer may be unreasonable. Consider a modification of Anna’s story: “I have 99 3/4 cups of dog food, and each dog at the shelter is fed 3 1/2 cups of food. How many dogs at the shelter can I feed?”

It would be reasonable for a student to answer this question with 28 dogs or 28 dogs with 1 3/4 cups of leftover food. However, since the appropriate unit is dogs, such a story could not realistically have a mixed-number solution. Yet when students apply algorithms for fraction division, they are faced with mixed-number results (i.e., 99 3/4 ÷ 3 1/2 = 28 1/2). Since 1/2 refers to one-half of a serving, not one-half of a dog, this could cause dissonance between the students’ algorithmic solution and their emerging conceptual understanding of fraction division. In such cases, students may need guidance to carefully select units and pose contexts in their problems so that mixed-number answers are sensible.

When students can anticipate how their problems will be answered, they can learn to conceptualize scenarios that better lend themselves to mixed-number answers. That is, to pose problems in which remaining amounts of one item can be realistically imagined as a fractional amount of another item (e.g., remaining children can be realistically imagined as a partial table; remaining cups of food can be imaginable as a partial day but not a partial dog).

2. Failing to Label Units

Many students did not use unit words in their story problems, creating uncertainty as to whether their stories involved division or multiplication. This may result from students’ thinking about fractions as operators, a definition that creates unitless fractions. For example, consider the following multiplication story: “I am painting my bedroom. On one wall, I used 2/3 of 4 1/2 quarts of paint. How much paint did I use?” Here, the operator fraction 2/3 has no unit. Brianna wrote, “I had 3 3/6 quarts of pop. I started on Wednesday and drank 2/6 each day. What day will I run out? How much will be leftover?” Although the questions she asked led in the direction of fraction division, her lack of unit designation for the 2/6 made her problem unclear. To
emphasize the division context, Brianna needed to indicate that she drank \( \frac{2}{6} \) quarts (of pop) each day. Without clear units, such stories can become difficult to interpret.

**Instructional recommendations:** Students are generally familiar with such whole-number problems as this: “I have 12 tomatoes and use 3 for each salad. How many salads can I make?” Even without labels, the “3” clearly refers to “3 tomatoes.” By paying special attention to the verbal and written language of classroom discussions, teachers can help learners understand the importance of designating unit words. In addition to reminding students to label units, discussion can clarify how fractions without units are understood to represent operators and indicate multiplication.

3. **Misusing Discrete Fractional Quantities**

Despite the fact that both discrete quantities (such as a number of bicycles) and continuous quantities (such as distance) can be correctly used in fraction division, nearly all students who successfully wrote a division of fractions word problem used continuous fractional quantities to represent measurements. On the other hand, problems that involved discrete objects were often unrealistic or failed to make sense. Katie was one of the few students who successfully applied the process of repeated subtraction to a set of discrete objects (see fig. 2).

Katie’s question (“How long will it take . . .?”) encapsulates a sophisticated shift from the usual discrete counting of days (“How many days will it take . . .?”), which is an important step in students being able to effectively handle discrete objects in fractional contexts. However, insight into Katie’s thinking about division of fractions was limited by her use of a whole-number divisor. Also, her knowledge of fractions in the real world is unclear because the scenario she created was unrealistic.

As shown in her ingenious table, Katie ignored the fractional amount of a butterfly remaining after she gave away her last full set. This may have stemmed from the way discrete objects like these can generate unreasonable fractional amounts (like 1/2 of a butterfly) or the difficulty created by her decision to give away butterflies every other day.

**Instructional recommendations:** “It is reasonable to assume that cognitive structures involving simple rational-number problems referring to a discrete model are different from those involved in solving rational-number problems referring to a continuous model” (Behr et al. 1983, p. 94). Teachers can bridge this gap by familiarizing students with continuous units during whole-number instruction (5 miles + 3 miles = 8 miles). Emphasizing unit words can help students think about the benefits of using the kinds of continuous items that can be realistically subdivided. Had Katie written about running 57 1/2 miles over the course of 3 months, her answer would have been less cumbersome. Also, natural extensions of this type of problem include rates (19 1/6 miles per month).

4. **Redirecting Fraction Contexts to Solve with Whole-Number Amounts**

Contexts that allow fractions to be redirected toward whole-number amounts can interfere with the necessity of solving a problem by dividing one fraction by another. For example, “Dylan has 3 3/4 packs of football cards, and each page protector in his..."
collector’s album can display 1/3 of a pack of cards. How many pages will he fill? Knowing that each pack contains 24 cards, Dylan could re-visualize this context as dividing 90 cards into groups of 8 for each page. Although a clever demonstration of number sense and a possible application of the quotient definition for fractions (90/8), such modifications can negate the opportunity to assess students’ understanding of division of fractions.

**Instructional recommendations:** It is exciting to see students demonstrate strong number sense by thinking ahead to create situations where fractions can be easily converted; however, these problems do not offer evidence of students’ conceptual knowledge of fraction division. Fraction-to-whole-number redirection was most consistently illustrated by students’ conceptualization of pizzas as a whole number set of slices, instead of a set of slices collectively representing a whole unit. Gabe demonstrated this issue in his problem: “If I had 2 1/4 pizzas and each pizza had 8 pieces in it, how many days could I eat pizza if I ate 3 pieces a day?”

Teachers may need to clarify the intended objective and pose problems that require division of fractional values. Doing so can help students develop realistic contexts in which fractions are not as easily transferred into whole numbers. Teachers can also discourage students from potential redirections by restricting conversions, such as 1 2/3 feet = 20 inches, in the problem-solving process.

### 5. Relying Disproportionately on Repeated Subtraction

We believe that students who selected non-whole-number divisors tended to apply repeated subtraction as their division process because it can be difficult to imagine sharing fairly across a non-whole-number set of groups. In fact, fewer than 10 percent of the seventh graders attempted to write fair-sharing stories. Only one student, Ethan, did so using a non-whole-number divisor (see fig. 3). Ethan was also the only student who used the familiar pizza context without converting his pizzas into slices, even though his divisor represented people. He creatively divided his 8 9/16 pizzas by a fractional amount (8 1/2) by suggesting, “One person just wanted half of what everyone else got.”

The rest of the students who posed fair-sharing problems divided fractional amounts by whole-number divisors. For example, Kinlee shared 6 2/3 pounds of salt between 2 people. Notice how Kinlee selected a dividend in which both the whole and fractional parts were easily divisible by her divisor, creating a problem that could be solved without applying a fraction-division algorithm (see fig. 4). Although Kinlee demonstrated sophisticated number sense, problems like these also make it difficult to assess a student’s understanding of division of fractions.

**Instructional recommendations:** Because so few students intuitively posed fair-sharing problems, whole-class discussions aimed at identifying, creating, illustrating, and solving meaningful fair-sharing problems involving fractions would be valuable. Fair sharing does not necessarily offer improvement over repeated subtraction. However, balanced instruction should incorporate both types of problems so that students experience division from multiple perspectives. Familiarity with both methods can help students generalize their knowledge of whole-number division toward a deeper understanding of dividing rational numbers.

Students often apply the standard multiply-by-the-reciprocal algorithm without having a conceptual awareness of why the algorithm works. Teachers can address this issue by posing and discussing a well-designed series of fair-sharing problems, moving from whole numbers to fractional values throughout. For example, one problem to pose can be borrowed from Kinlee’s salt context: “Kinlee evenly sprinkles 2/3 of a teaspoon of salt across 1 1/4 bowls of popcorn. How much salt would she need for just one bowl of popcorn?”

By capturing the solution steps for Kinlee’s modified problem, both pictorially and symbolically (see fig. 4), teachers can illuminate the connection between the fair-sharing process and the multiply-by-the-reciprocal algorithm (Gregg and Gregg 2007; Siebert 2002). The fair-sharing
Fig. 4 Teachers can connect the fair-sharing process and the multiply-by-the-reciprocal algorithm by modeling how Kinlee shared, both pictorially and symbolically, how 6 2/3 pounds of salt were divided between 2 people.

2/3 = 10/15 teaspoons of salt redistributed into 5 regions. Each region is 1/4 of a bowl of popcorn.

1. 10/5 teaspoons ÷ 5 regions = 2/15 teaspoons per region
2. Since there are 4 regions per bowl, 2/15 teaspoon × 4 regions per bowl = 8/15 teaspoons per bowl

(a) Using pictures

The fair-sharing algorithm:

\[
\frac{A}{B} + \frac{C}{D} = \frac{AC}{BC} + \frac{C}{D}
\]

Step 1: \(\frac{AC}{BC} + C = \frac{A}{BC}\)

Step 2: \(\frac{A}{BC} \cdot D = \frac{AD}{BC}\)

The multiply-by-the-reciprocal algorithm:

\[
\frac{A}{B} + \frac{C}{D} = \frac{A}{B} \cdot \frac{D}{C} = \frac{AD}{BC}
\]

(b) Using symbols
using continuous rather than discrete measurements, labeling unit words consistently, emphasizing the relationship between remainders and mixed-number answers, and encouraging students to use multiple definitions of fractions (especially operator) and division (especially fair sharing). It is also important to address students’ selections of incorrect operations when trying to pose division of fraction problems by listening to them talk about the reasoning behind their operation choices. Through carefully sequenced instruction and targeted discussions, both whole-class and individual, teachers can prevent and help students overcome limited conceptions that can interfere with their ability to conceptually understand fraction division.

REFERENCES

Any thoughts on this article? Send an e-mail to mtms@nctm.org.—Ed.